Finding Meteorites with Mathematics

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Everyone knows that the equation

$$ax + by + c = 0 \tag{1}$$

represents a straight line. But what if you have a three-dimensional problem? Presumably a straight line in three dimensions is represented by the equation

$$ax + by + cz + d = 0. \tag{2}$$

Wrong! Equation (2) is the equation, in three dimensions, of a *plane*. To represent a *line* in three dimensions, we need *two* equations of the form of Equation (2); a line is the intersection of two planes and we need to give the equations of two planes to specify it.

Three non-collinear points ought to define a plane, so, if we know the coordinates of three points, (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), how can we determine the equation of the plane containing these three points? Well, we can write down the equations

$$ax_1 + by_1 + cz_1 + d = 0, \tag{3}$$

$$ax_2 + by_2 + cz_2 + d = 0, \tag{4}$$

and

$$ax_3 + by_3 + cz_3 + d = 0. \tag{5}$$

That doesn't seem to be enough to solve for the four unknowns, a, b, c and d. If I want a condition for a point (x, y, z) to be in the plane, I combine Equation (2) with Equations (3), (4), and (5) to give me four equations in the three unknowns,

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{6}$$

This system has a solution if and only if the matrix has a zero determinant, that is,

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0, \tag{7}$$

and (7) is the equation of the plane containing the three points.

Having disposed of that little bit of mathematics, we can now get down to the topic announced in the title of this article.

Until recently, I served on the Meteorites and Impacts Advisory Committee to the Canadian Space Agency. One of our duties was to investigate reports of *fireballs* and to see if we could find an associated meteorite. Most of the faint *shooting stars* or *meteors* that we can see in the sky on any night are tiny particles of dust of cometary origin. Occasionally, however, a much more spectacular phenomenon is reported. A huge ball of light streaks across the sky, illuminating the countryside for hundreds of miles around, perhaps accompanied by thunderous noise, attracting widespread public attention or even alarm. This is a *fireball*, and it is a large chunk of stone or iron of asteroidal origin. While in orbit around the Sun, it was a *meteoroid*. If any of it survives the fiery plunge through the atmosphere, the specimen that reaches the ground is a *meteorite*. Figure 1 shows a fragment of the Canyon Diablo meteorite that fell in Arizona about 50,000 years ago. The speed of the fireball is several tens of kilometres per second and its path through the atmosphere, which lasts for just a few seconds, is nearly a straight line. I wondered if I could use a little mathematics to help track the fireball through the atmosphere.

I found that talking to eyewitnesses by telephone was interesting, but did not produce much in the way of quantitative information. So, I bought myself a compass and a clinometer (the latter measures angular height above the horizon) and I decided that the best way to proceed was to interview each witness *in situ*, within a few days at most from the event. You ask the witness to re-enact exactly what he or she was doing when the fireball appeared and to point to two points on its track through the sky. The directions to these two points together with the geographical position of the witness are sufficient to define a plane that contains the path of the fireball. Then you visit another witness, maybe 50 km or so away, and ask him or her to indicate the directions to two points. This gives a second plane, and, where the planes intersect is the path of the fireball.

One thing that investigators commonly find is that witnesses very commonly believe that the object they saw was only a few hundred yards away and many of them will swear that they saw it land in the next field. In reality, the object is several tens or even hundreds of kilometres away.

Here's how the geometry works. In the first place I assume a Flat Earth. This is not because I am a member of the Flat Earth Society. It is justified (within the limited precision of eyewitness accounts) by the circumstances that the height of a witness above sea level is very much smaller than the height of the fireball above the ground, and the height of the fireball above the ground is very much smaller than the radius of Earth. I set up a rectangular coordinate system with the origin at some arbitrary point on Earth (usually for convenience

Figure 1: A fragment of the Canyon Diablo iron meteorite from the author's collection.

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a little bit southwest of all witnesses), with the x axis pointing east, the y axis north, and the z axis straight up. My southwest choice of origin means that all witnesses are in the first quadrant, so I don't have to deal with minus signs!

For each witness I record his or her x and y coordinates, and I measure the spherical coordinates θ and ϕ of two points on the sky track. These angles are, respectively, the angular distance of a point from the zenith, and the azimuth or bearing, measured counterclockwise from the x axis (east) in the usual manner for spherical coordinates. If you refer to Figure 2 you will see that the fireball is in a plane containing the following three points:

$$W = (x_0, y_0, 0), \tag{8}$$

$$A = (x_0 + r_1 \sin\theta_1 \cos\phi_1, y_0 + r_1 \sin\theta_1 \sin\phi_1, r_1 \cos\theta_1), \tag{9}$$

and

$$B = (x_0 + r_2 \sin\theta_2 \cos\phi_2, y_0 + r_2 \sin\theta_2 \sin\phi_2, r_2 \cos\theta_2). \tag{10}$$

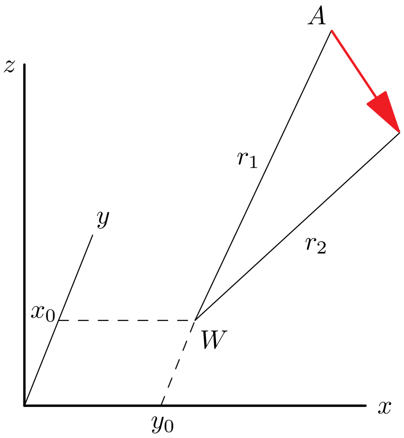

Figure 2: A witness at W sees a fireball go from A to B. The spherical coordinate angles θ and ϕ are measured for both points. What is the equation of the plane WAB?

Of course we don't know either of the distances r_1 or r_2, but if you use Equation (7) to find the equation of the plane containing the path of the fireball, you'll find (as you would expect) that it doesn't depend on these distances.

Then, as mentioned, we interview a second witness, and define a second plane. These two planes define the path of the fireball, which in practice is essentially a straight line. In practice we have many witnesses, all inconsistent with each other, and we have to handle that as best we can!

Perhaps a numerical example is in order. I give below the coordinates (in kilometres) of two hypothetical observers and the spherical coordinates (in degrees) of two points on the sky track as seen by each of the witnesses:

x_0	y_0	θ_1	ϕ_1	θ_2	ϕ_2
15	5	25.5	54.5	36.7	16.7
30	15	29.5	202.9	33.6	242.9

Perhaps you can use these data to calculate the two planes whose intersection gives you the atmospheric trajectory of the fireball. I obtain

$$0.1260x + 0.3162y - 0.1577z - 3.4712 = 0 \tag{11}$$

and

$$0.2683x + 0.1598y + 0.1757z - 10.4445 = 0. \tag{12}$$

These two planes, then, define the path—but can you visualize it? You will be able to visualize it if

(a) you can calculate the point where the path intersects the ground;

(b) you can determine the *ground track*, which is the vertical projection of the atmospheric track on the ground;

(c) you can determine the angle that the path makes with the vertical.

For (a), just put $z = 0$ in each of the equations. This will give you two equations in x and y. Solve for these and they give you the coordinates of the *extrapolated ground-level point*. Can you perform this calculation? I compute $(42.4, -6.0)$; that is, 42.4 km east and 6.0 km south of your origin. Usually the meteorite will fall somewhat short of this point.

For (b), eliminate z between the two equations. This will give you a single equation in x and y, and this is the equation of the ground track. I find

$$y = -0.79796x + 35.02; \tag{13}$$

the meteorite lies somewhere along this line, and a little before the extrapolated ground-level point.

For (c), you already know the coordinates of one point (the extrapolated ground level point) on the path. Calculate the coordinates of any other point on the path, and this will enable you to determine the angle it makes with the vertical. This I leave to you.

So far I haven't managed the ultimate goal of finding the prize—a meteorite—from the relatively crude angle estimates made by eyewitnesses, although four meteorites have been recovered from the much more precise measurements that can be made from photographic records. A photograph allows very precise measurements to be made with a microscope, but we then have to be much less cavalier with the mathematics. A Flat Earth approximation just won't do! Figures 3 and 4 show two photographs, obtained by Barry Burgess and Michael Boschat on November 19, 2002, from two sites in Nova Scotia about 45 km apart. You can see that the starry background is different in the two photographs. From measurements we were able to calculate that the height of the

Figure 3: A meteor photographed from Nova Scotia by Barry Burgess.

Figure 4: The same meteor as is shown in Figure 3. The point of observation was 45 km away from that of Figure 3. Photograph taken by Michael Boschat.

meteor was 112.16 km when it was first detected, with an error of only 20 metres.

Another interesting aspect to this work is that the flight of a fireball through the atmosphere is often accompanied by thunderous noise. The fireball is usually so high in the atmosphere that the sound may take several minutes to reach the ground, and an eyewitness may not always associate the noise with the fireball that was seen. What is exciting is that it has been recognized in recent years that sound from a fireball can be detected by seismographs, and these can record the exact time of arrival of the sound signal, thereby giving scope for more mathematics. Figure 5 shows a seismic record of a fireball.

When a meteoroid streaks through the atmosphere at many times the speed of sound, it generates a conical shock front of very small (often less than a degree) semivertical angle, and this shock front can be recorded on seismographs. During this stage of the flight through the atmosphere, the surface of the meteorite becomes extremely hot, and much of the surface vaporizes. The flight through the atmosphere during this supersonic phase lasts only a few seconds, and there isn't much time for the heat to penetrate to the interior. Because of this, a tremendous temperature gradient and consequent thermal stress may be set up in the stone, and it may suddenly disintegrate in a violent, explosive *terminal burst*. This generates another shock front (initially spherical), which can also be detected by seismographs. If there is no violent terminal burst (iron meteorites are stronger than stony meteorites), a substantial chunk may survive. It will slow down to a speed low enough that its glow can no longer been seen (this probably happens while its speed is still supersonic, though it will eventually reach subsonic speed), and it may subsequently fall to Earth as a relatively cold stone. (We often get reports of meteorites landing and starting a fire, but it is very doubtful whether this ever happens!) This slow fall may take a few minutes, and the path is no longer a straight line, which is why it will fall short of the extrapolated ground-level point. The impact may also be heard by seismographs. In that case, the sound travels through the ground, much faster than through the air, so that a seismograph may record the impact first, and the atmospheric shock fronts later, which can be confusing. A meteorite typically has a thin black fusion crust to indicate how the surface (but not the interior) has been subject to great heat.

The entire seismic phenomena can be quite complicated because of the three separate events, so, for the purpose of this article, let's keep it simple and concentrate just on the *terminal burst*, which we regard as a point source of sound somewhere in the atmosphere. It generates a spherical shock front, which is heard at a number of seismographic stations. If it is recorded at four stations, it should be possible to find the position (x_0, y_0, z_0) and the time t_0 of the explosion. This is not too hard, because at time t the radius of the spherical shock front will be $v(t - t_0)$, where v is the speed of sound. The equation of the spherical shock front at time t is therefore just

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = v^2(t - t_0)^2. \tag{14}$$

If you know the positions (x, y, z) of four stations and the arrival times t at each of them, you can set up four equations like this, and solve them for (x_0, y_0, z_0, t_0). The four equations are each quadratic in the four unknowns, but, provided you know how to solve four simultaneous quadratic equations in four unknowns (!), there will be no difficulty. (If there are more than four stations, we have to perform a least squares solution, a technique from statistics.)

An example is in order. Let's suppose that the coordinates of the four stations in kilometres and the arrival times in seconds are

x	y	z	t
15	4	0.1	92
5	38	0.3	81
36	20	0.4	82
20	33	0.5	59

Suppose that the speed of sound is 0.33 km s^{-1}. Can you set up the four equations and then solve them? Quite a challenge! I make out the answer to be $x_0 = 18.50$ km, $y_0 = 26.06$ km, $z_0 = 14.45$ km, and $t_0 = 11.55$ s.

That was relatively painless, because we made the assumption that the temperature of the atmosphere (and hence the speed of sound) is the same at all heights, and consequently sound travels in straight lines. But this is far from the case in the real atmosphere, and sound does not travel in straight lines. I therefore looked in several textbooks, and they told me that the path of a sound wave in the atmosphere is an arc of a circle. This was promising information, but I needed to know exactly how big a circle, and where the centre was, so I needed to try and prove for myself that the path is a circle. This took me a little while, but I eventually managed it by making the assumption that the speed of sound in the atmosphere decreases linearly with height. Then indeed, sound rays are arcs of circles.

I should have been pleased with this—but in fact I was puzzled. I knew that in the lower 11 km or so of the atmosphere—the part known as the *troposphere*—the *temperature*, to a good approximation falls off linearly with height at a rate of about 6.5 °C/km, which is called the *temperature lapse rate*. (Above 11 km, in the *stratosphere*, the lapse rate changes.) Since the speed of sound depends on the square root of the temperature, the sound speed also falls off as the square root of the height, not linearly. After struggling with that for a while, I found that the path of a

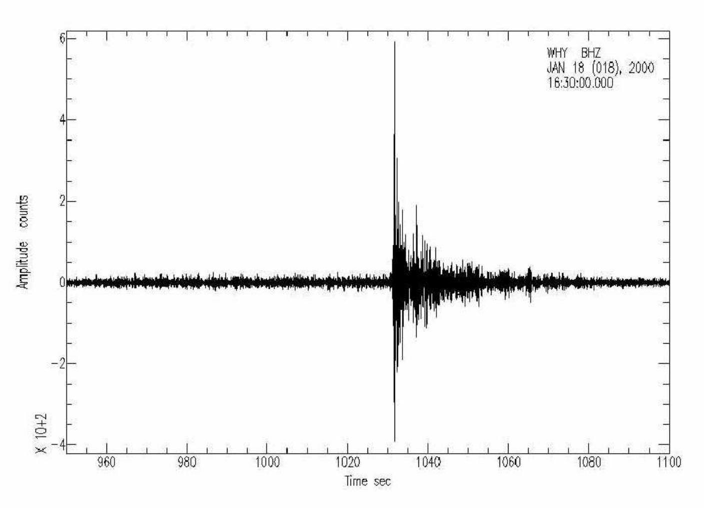

Figure 5: Seismogram of the terminal burst of a fireball. This was obtained at the Whitehorse (Yukon) seismic station on 18th January, 2000. The height of the terminal burst was about 37 km, and the seismic station was distance about 30 km from the sub-burst point. A meteorite from this fireball was subsequently recovered from the frozen surface of Tagish Lake in the extreme northwest corner of British Columbia, and turned out to be one of the most primitive meteorites known, very little changed from the early stages of the formation of the solar system. The seismogram was kindly supplied by the Geological Survey of Canada, courtesy of Dr. John Cassidy, Pacific Geoscience Centre.

sound wave in the atmosphere is not along an arc of a circle at all—it is along the arc of a curve I remember studying in math class long ago, namely a *cycloid*! So, apparently you can't necessarily believe everything you read in books all of the time—not even scientific ones!

However, the geometry of the cycloid, while great fun, is not quite so easy as that of a circle, so, in order to prevent this article from becoming too complicated, let's go back to the supposition that the sound speed falls off linearly with height. This will not change any major conclusions, though the details may not be quite accurate. Specifically, we'll suppose that the sound speed v at height z is given by

$$v = v_0 - kz. \tag{15}$$

Here v_0 is the sound speed at ground level, and k is a constant that shows how fast the sound speed decreases with height. What I found was that, if a sound wave at any given level makes an angle ψ with the horizontal, it subsequently moves along the path

$$\left(x - \frac{v_0 \tan\psi}{k}\right)^2 + \left(z - \frac{v_0}{k}\right)^2 = \left(\frac{v_0 \sec\psi}{k}\right)^2. \tag{16}$$

You will probably recognize this as a circle, and you can probably say what its radius is, and where the centre is. If we express the horizontal distance x and height z in units of v_0/k, the equation looks a little easier:

$$(x - \tan\psi)^2 + (z-1)^2 = \sec^2\psi. \tag{17}$$

Now let us imagine that a terminal burst takes place when a meteoroid is at a horizontal distance $x = 1.5$ from some origin on the ground, and at a height $z = 0.5$ above ground level. Sound is emitted in all directions, and, in Figure 6 we see the paths of several sound rays at different starting angles ψ_0. The continuous curves are for $\psi_0 = 30°, 40°, 50°, 60°, 70°$ and $80°$. The dashed path is for $\psi_0 = 48.2°$ and it is seen that it just scrapes the ground at $x = 0.38$. Anyone to the left of this position—i.e. anyone more distant than 1.12 from the sub-burst point—will not hear the burst. This is not because he or she is too far from the explosion, but because the sound never reaches the ground; it moves instead in a circular arc.

Of course calculating the atmospheric trajectory from the signal arrival times now becomes much more complicated if the speed of sound varies with height and if the sound moves in arcs of circles, or, worse, of cycloids, but the principles are the same, and you just have to fill a few more sheets of paper with equations and tear out a few more handfuls of hair. We haven't yet actually recovered a meteorite from seismograph records, but, mark my words—we will, we will!

In a previous article in this magazine I showed how mathematics is useful if you are interested in moths. This time I have shown that mathematics is useful if your interest is meteorites. It seems that, however obscure or esoteric one's interests, mathematics seems always to have a role to play.

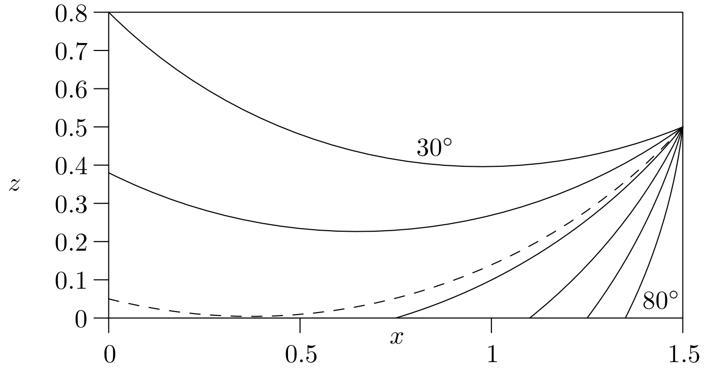

Figure 6: Sound from an explosion at the right of the figure travels in circular paths and never reaches the ground to the left of $x = 0.38$.

References

Much of the material in this article has been adapted from a number of previously-published technical papers by the author, as follows.

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